Equivariant Bordism and Thom Spectra

Def. (Gepner–Henriques)

Glo: ∞ cat w. – Objects: Compact Lie gps
– $Map_{Glo}(H,G) \simeq Hom(H,G)//G \simeq \coprod_{\substack{\alpha: H \to G \\ conj. cl.}} BC(\alpha)$.

$Orb \subseteq Glo$ wide subcat. on injective gp hom.

For G cpct Lie,

$$Orb_{/G} \xrightarrow{\sim} Orb_G$$
$$H \to G \longmapsto G/H$$

$$\mathcal{S}^{glo} := PSh(Glo) \xrightarrow{res_G} \mathcal{S}^G := PSh(Orb_G).$$

Ex: $BO(n) := Map_{Glo}(-, O(n)) \in \mathcal{S}^{gl}$

$res_G BO(n)$ classifies G-equiv. rk n vector bundles:

$$N_\Delta\left(\begin{array}{c}\text{top. category of rk } n \text{ } G\text{-eq. vbds} \\ \text{over } G\text{-CW cmplxs}\end{array}\right) \simeq \mathcal{S}^G/res_G BO(n).$$

obj: $E \to X$ G-equiv rk n vbdl, X has G-CW str.

$Hom\left(\begin{array}{c}E \\ \downarrow \\ X\end{array}, \begin{array}{c}E' \\ \downarrow \\ X'\end{array}\right) = $ G-equiv bundle maps $\subseteq \mathcal{C}(E, E')$.

$O(n) \to O(n+1)$

$b_G O := res_G \operatorname{colim}_n BO(n)$ classifying space for stable (wrt $+\mathbb{R}$) G-equiv. vb.

Def. $\theta \to b_G O \in \mathcal{S}^G$.

M smooth G-mfd. $c_{TM}: M \to b_G O$ classifying map of tangent bundle

$$\left\{ \begin{array}{ccc} & & \Theta \\ & \nearrow & \downarrow \\ M & \xrightarrow[c_{TM}]{} & b_G O \end{array} \right\} = \pi_0 \mathrm{Map}_{\mathcal{S}^G/b_G O}(c_{TM}, \Theta) = \Theta_G(M)$$

G-equivariant Θ-orient. of M

Examples:

$b_G SO := \mathrm{res}_G \operatorname{colim}_n BSO(n)$ Orientations of M st. G acts orientation-preserving

$b_G U := \mathrm{res}_G \operatorname{colim}_n BU(n)$ ⇝ G-equiv. complex str on $T_x M \oplus \mathbb{R}^k$

$*$ = G-equiv. stable framing ($T_x M \oplus \mathbb{R}^k \simeq \mathbb{R}^{k+|M|} \times M$.)

B smooth G-manifold w. boundary ∂B.

$$T_x B|_{\partial B} \simeq T_x \partial B \oplus \mathbb{R}$$

($\simeq$: inner normal trivialis.)

$c_{TB|\partial B} \simeq c_{T\partial B}$

induces $\partial_{in} : \Theta_G(B) \longrightarrow \Theta_G(\partial B)$.

Def. (Θ-oriented bordism)

X topological G-space.

$$\mathcal{N}^{G,\Theta}_*(X) := \left(\{(M, o, f: M \to X)\}/\cong, \sqcup \right) \Big/ (\partial B, \partial_{in} o_B, F|_{\partial B})$$

(M: smooth cpt G mfd; $o \in \Theta_G(M)$; B: compact smooth G mfd)

$\mathcal{N}^{G,\Theta}_*(-) : \mathrm{Top}^G \longrightarrow \mathrm{grAb}$ upgrades to a homology theory for top. G-spaces.

Rk (Tangential versus normal structures)

$b_G O \in CMon(\mathcal{S}^G)$ (via direct sum of G-represent.)

$b_G O \xrightarrow{i} B_G O =: b_G O^{gp}$ group completion

$B_G O$ classifies stable (wrt $+V$) equiv. vector bundles.

$$M \xrightarrow[c_{TM}]{} b_G O \xrightarrow{i} B_G O \xrightarrow[(-)^{-1}]{} B_G O$$

classifying map for stable normal bundle.

II Equivariant Thom spectra via parametrized htpy theory

$Cat_G := Fun(Orb_G^{op}, Cat)$.

Examples: $\mathcal{S}^G \subseteq Cat_G$

$\underline{\mathcal{S}}^G : G/H \longmapsto \mathcal{S}^H$ free G-cocomplete cat on $*$

$X \in \mathcal{S}^G$ $\underline{\mathcal{S}^G_{/X}} : G/H \longmapsto \mathcal{S}^H/res^G_H X$ free G-cocomplete cat. on X

$\underline{Sp}^G : G/H \longmapsto Sp^H$ free representation-stable G-cocomplete cat.

$\underline{Rep} := \coprod_n res_G(BO(n)) = G/H \longmapsto H\text{-rep}.$

Construction (∂-homomorphism)

$$\exists\quad \underline{\mathrm{Rep}} \xrightarrow{\ \mathrm{sph}(-)\ } \underline{\mathcal{S}}^G_* \in \mathrm{Fun}(\mathrm{Orb}_G^{op}, \mathrm{CMon}(\mathcal{S})) \cong \mathrm{CMon}(\mathrm{Cat}_G)$$

$$V \longmapsto S^V$$

$$\begin{array}{ccccc}
\underline{\mathrm{Rep}} & \xrightarrow{\mathrm{sph}(-)} & \underline{\mathcal{S}}^G_* & \xrightarrow{\Sigma^\infty} & \underline{\mathrm{Sp}}^G \\
\downarrow & & & & \cup\!\! \mid \\
\underline{\mathrm{Rep}}^{gp} & & \xrightarrow{\quad \partial \quad} & & \underline{\mathrm{pic}}(\mathrm{Sp}^G) \\
\cong & & & & \\
\underline{\mathrm{Box}}_G \times \mathbb{Z} & & & &
\end{array}$$

Def. (Thom spectrum functor).

$$\begin{array}{ccc}
\underline{\mathrm{Rep}}^{gp} & \xrightarrow{\quad \partial \quad} & \underline{\mathrm{Sp}}^G \\
\downarrow & \nearrow & \\
\underline{\mathcal{S}^G/\mathrm{Rep}^{gp}} & &
\end{array}$$

Shah/Martini-Wolf:
$\exists!$ G-cocontinuous, symmetric monoidal functor $=:$ Th

Thom spectrum functor

$$\left(\mathrm{Th}^H : \mathcal{S}^H/\mathrm{Rep}^{gp} \longrightarrow \mathrm{Sp}^H \right)_{H \leq G} \in \mathrm{CAlg}(\mathrm{Pr}^L)$$

and for all $H \leq K$ closed subgps of G,

$$\begin{array}{ccc}
\mathcal{S}^K/\mathrm{Rep}^{gp} & \xrightarrow{\ \mathrm{Th}^K\ } & \mathrm{Sp}^K \\
K\times_H - \uparrow \dashv \downarrow \mathrm{res}^K_H & \Leftarrow & \downarrow \mathrm{res}^K_H \vdash \uparrow K_+ \wedge_H - \\
\mathcal{S}^H/\mathrm{Rep}^{gp} & \xrightarrow{\ \mathrm{Th}^H\ } & \mathrm{Sp}^H
\end{array}$$

i.e. $K_+ \wedge_H \mathrm{Th}^H \simeq \mathrm{Th}^K(K \times_H -)$.

Rk: Analogously, $\exists!$ G-cocontinuous (symm. monoidal)

$$\begin{array}{ccc} \mathcal{S}^G/_{\underline{\mathrm{Rep}}} & \overset{th}{\dashrightarrow} & \mathcal{S}^G_* \\ \uparrow & \nearrow_{\mathrm{Sph}(-)} & \\ \underline{\mathrm{Rep}} & & \end{array}$$

Thom space functor

$$\begin{array}{ccc} \mathcal{S}^G/_{\underline{\mathrm{Rep}}} & \xrightarrow{th} & \underline{\mathcal{S}}^G_* \\ \downarrow & & \downarrow \Sigma^\infty \\ \mathcal{S}^G/_{\underline{\mathrm{Rep}}^{gp}} & \xrightarrow{Th} & \underline{Sp}^G \end{array} \quad \in \mathrm{CAlg}(\mathrm{Pr}^L_G)$$

Gepner-Nikolaus-Schwede

$$\mathrm{Rep}^{gp} \simeq B_G O \times \mathbb{Z} \Leftarrow B_G O \overset{i}{\Leftarrow} b_G O$$

Def. $m: \mathcal{S}^G/_{b_G O} \xrightarrow{-i} \mathcal{S}^G/_{\mathrm{Rep}^{gp}} \xrightarrow{Th^G} Sp^G$

Choose sign s.t. $m(c_{TM}) = \Omega^{V-\dim M} Th(\nu_V(M))$.

$M \hookrightarrow V$ — M smooth G-mfd, V fd. G-rep.

⇝ Can construct Thom-Pontryagin map

$$\Psi^G: \mathcal{N}^{G,\Theta}_*(-) \longrightarrow m\Theta^G_*(-).$$

Theorem (Thom, Pontryagin, Wasserman, tom Dieck, Schwede, B.)

1) If the connected component of the identity $G^\circ \subseteq G$ is central, then Ψ^G is an isomorphism.

2) If $G^\circ \subseteq G$ is not central and $\Theta_G(*) \neq \emptyset$, then $\mathcal{N}^{G,\Theta}_*(-)$ is NOT represented by a genuine G-spectrum.

Ingredients for the proof:

① Thom-Pontryagin

② Wirthmüller homomorphisms:

$H \leq G$. $\quad Sp^G \underset{\text{coind}^G_H}{\overset{\text{ind}^G_H}{\rightleftarrows}} Sp^H$ (with $\text{ind}^G_H \dashv \text{res}^G_H \dashv \text{coind}^G_H$), $\quad \text{ind}^G_H \simeq \text{coind}^G_H(- \otimes S^{T_eG/H})$.

$$\Rightarrow \text{Wirth}^G_H : m\Theta^G_*(G/H) \xrightarrow{\simeq} m\Theta^H_*(S^{T_eG/H})$$

$$\begin{array}{ccc} m\Theta^G_*(G/H) & \xrightarrow{\simeq} & m\Theta^H_*(S^{T_eG/H}) \\ \uparrow \psi^G & \circlearrowleft & \uparrow \psi^H \\ \mathcal{N}^{G,\Theta}(G/H) & \xrightarrow{\text{Wirth}^G_H} & \mathcal{N}^{H,\Theta}(S^{T_eG/H}) \\ (M, o, f: M \to G/H) & \longmapsto & (M, o, q \circ f) \end{array}$$

construct

$q: G/H_+ \longrightarrow S^{T_eG/H}$
collapse map induced
by tubular nbhd of $H \subseteq G$.

Lemma If $H \curvearrowright T_eG/H$ trivial, Wirth^G_H is an iso.

③ ψ^G is an isomorphism on geometric fixed-points.

$$\begin{array}{ccc} \mathcal{S}^G & \xrightarrow{(-)^G} & \mathcal{S} \\ \downarrow & & \downarrow \\ Sp^G & \dashrightarrow & Sp \end{array} \quad \exists!\ \Phi^G$$

Proposition: $\widetilde{\mathcal{N}}^{G,\Theta}_*(\widetilde{E}\mathcal{P}) \xrightarrow{\simeq} m\Theta^G_*(\widetilde{E}\mathcal{P}_+)$

Proof idea: Geometric fixed points of Thom spectra are Thom spectra: $\in \text{CAlg}(\Pr^L)$

$$(b_G O)^G = \coprod_{j \in \mathbb{N}_0} bO \times Gr_j^{G,+}$$

$$\Rightarrow \Theta^G = \coprod_{j \in \mathbb{N}_0} \underbrace{(\Theta^G \times_{b_G O^G} bO \times Gr_j^{G,+})}_{=: \Theta(j)}$$

$$\Phi^G(m\Theta) = \bigoplus_{j \in \mathbb{N}_0} \Sigma^j m(\Theta(j)).$$

↖ non-equiv. Thom spectrum.

$$m\Theta^G_*(\tilde{E}P_+) \cong \bigoplus_{j\in\mathbb{N}_0} m\Theta^{(j)}_{*-j}$$

$$\uparrow \psi^G \qquad\qquad \cong\uparrow \psi^e$$

Same splitting for bordism:

$$\tilde{\mathcal{W}}^{G,\Theta}_*(\tilde{E}P_+) \cong \bigoplus_{j\in\mathbb{N}_0} \tilde{\mathcal{W}}^{\Theta(j)}_{*-j}(S^0) = \mathcal{W}^{\Theta(j)}_{*-1}(*)$$

$$(M,o,f) \longmapsto (M^G, o_{|M^G}, f_{|M^G})$$

$$M^G \xrightarrow{c_{TM}} b_G O^G \xrightarrow{\text{project}} bO, \qquad c_{T(M^G)} : M^G \to bO$$

Proof of main thm

① Induction on $\dim G$, $\# \pi_0 G$.

Induction hypothesis + Wirthmüller isomorphism $\Big\} \Rightarrow \psi^G(G/H)$ is iso $\forall H \subsetneq G$.

$\Rightarrow \tilde{\mathcal{W}}^{G,\Theta}(X_+) \simeq m\Theta^G_*(X)$ for all topol. G-spaces X with $X^G = \emptyset$.

cofiber sequence $EP \to S^0 \to \tilde{E}P_+$ + geometric fixed pts

$EP^H = \begin{cases} \emptyset & H = G \\ * & H \subsetneq G \end{cases}$

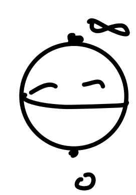
$\Rightarrow \mathcal{W}^{G,\Theta}(*) \simeq m\Theta^G(*)$. "proves" 1)

②: If $G^0 \subseteq G$ not central, $\exists H \subseteq G$ s.t. $H \circlearrowright T_e G/H$ non-trivial.

↯ $\mathcal{W}^{G,\Theta}_*(G/H) = 0$ for $* < \dim G/H$

$T_e G/H = V \oplus \mathbb{R}^k$, $V^H = 0$

$\tilde{\mathcal{W}}^{H,\Theta}_k(S^{T_e G/H}) \cong \tilde{\mathcal{W}}^{H,\Theta}_0(S^V) \neq 0$ ↖ $\Theta_H(*) \neq \emptyset$ □



Multiplicative structures

The Thom spectrum functor takes group completions to localizations of ring spectra. $\Theta \to b_G O \in \mathrm{CMon}(\mathcal{S}^G)$

Def. (Inverse Thom class)

V G-representation, $o \in \Theta_G(S^V)$ Θ-orient. $(S^V, o, id_{S^V}) \in \tilde{\mathcal{N}}^{G,\Theta}_{\dim V}(S^V)$

$$\begin{array}{ccc} \mathbb{I} & & \downarrow \psi^G \\ \tau_{V,o} & & m\Theta^G_{\dim V}(S^V) \\ \| & & \end{array}$$

inverse Thom class.

Proposition (Schwede, B.)

For $\Theta \to bO \in \mathrm{CMon}(\mathcal{S}^G)$,

$Th^G(\Theta) \longrightarrow Th^G(\Theta^{gp})$ is localization at the inverse Thom classes $\tau_{V,o}$ V G-represent, $o \in \Theta_G(S^V)$.

In particular, for $X \in \mathrm{Top}^G$,

$$Th(\Theta^{gp})^G_*(X) = Th(\Theta)^G_*(X)[\tau_{V,o}^{-1}]$$

$$= \operatorname*{colim}_{(o,V)} \left(Th(\Theta)^G_*(X) \xrightarrow{\cdot \tau_{V,o}} Th(\Theta)^G_*(X \wedge S^V) \to \dots \right)$$

We can perform the same localisation on bordism

Def: $\mathcal{N}^{G,\Theta}_* := \mathcal{N}^{G,\Theta}_*(-)[\tau_{V,o}^{-1} \mid V\ G \text{ rep}, o \in \Theta_G(S^V)]$

"stable / homotopical bordism".

Obtain:

$$\begin{array}{ccc} \mathcal{N}^{G,\Theta} & \xrightarrow{\psi^G} & m\Theta^G_*(-) \\ \downarrow & & \downarrow \\ \mathcal{N}^{G,\Theta} & \xrightarrow{\overline{\psi^G}} & Th(\Theta^{gp})^G_*(-) \end{array}$$

Proposition (Schwede, B.)

Suppose that for all $H \leq G$, $\exists$ H-rep. V st.
$$\Theta_H(S^V) \neq \emptyset \text{ and } \Theta_H(S^{T_e G/H \otimes V}) \neq \emptyset.$$

Then
$$\eta^{G,\Theta} \xrightarrow[\psi^G]{\simeq} \mathrm{Th}(\Theta^{gp})^G_*$$

Global Thom spectra

$\mathrm{Cat}_{\mathrm{Glo}} := \mathrm{Fun}(\mathrm{Glo}^{op}, \mathrm{Cat})$.

$\underline{S}^{gl} : \mathrm{Glo}^{op} \longrightarrow \mathrm{Cat} \qquad G \longmapsto S^{gl}_{/BG} = G\text{-global spaces}.$

$\underline{S}^{orb} : \mathrm{Glo}^{op} \longrightarrow \mathrm{Cat}, \qquad G \longmapsto S^G$

$\underline{S}^{orb} \subseteq \underline{S}^{gl}$ fully faithful.

$$\exists\ \mathrm{Sph}(-) : \underline{\mathrm{Rep}} \longrightarrow \underline{S}^{orb}_* \subseteq \underline{S}^{Glo}_*, \qquad V \longmapsto S^V \qquad \in \mathrm{CMon}(\mathrm{Cat}_{\mathrm{Glo}})$$

Extends uniquely to a Glo-cocontinuous symmetric monoidal

$$\mathrm{th} : \underline{S}^{gl}_{/\underline{\mathrm{Rep}}} \overset{\exists!}{\dashrightarrow} \underline{S}^{Glo}_* \quad \text{Thom space fctr.}$$

Proposition/Definition (Gepner-Nikolaus-Schwede / Lenz-B.)

$$\begin{array}{ccc} \underline{S}^{gl}_{/\underline{\mathrm{Rep}}} & \xrightarrow{\mathrm{th}} & \underline{S}^{gl}_* \\ \downarrow & & \downarrow \Sigma^\infty \\ \underline{S}^{gl}_{/\underline{\mathrm{Rep}}^{gp}} & \xrightarrow{\mathrm{Th}} & \underline{Sp}^{gl} \end{array} \quad \text{in } \mathrm{CAlg}(\mathrm{Pr}^L_{\mathrm{Glo}}).$$

Thom spectrum functor

Glo-category on G-global spectra

On global sections, we get

$$S^{gl}_{/\mathrm{Rep}^{gp}} \xrightarrow{\mathrm{Th}^{gl}} Sp^{gl} \in \mathrm{CAlg}(\mathrm{Pr}^L).$$

$$\begin{array}{ccc} \mathcal{S}^{gl}/_{Rep} & \xrightarrow{Th^{gl}} & Sp^{gl} \\ \downarrow res^G & \swarrow\!\!\sim & \downarrow res_G \\ \mathcal{S}^G/_{Rep^{gp}} & \xrightarrow{Th^G} & Sp^G \end{array}$$

<u>Upshot</u>: The classical equivariant Thom spectra all come from global spectra. mO, MO, mU, MU, mSO, MSO, . . .

This has been applied for computations.

E.g.: <u>Thm</u> (Hausmann) For G abelian,

$$MU^G := Th^G(\underbrace{B_G U}_{=: res_G (\operatorname{colim}_n BU(n))^{gp}})$$

carries the universal G-equivariant formal gp law.

Hausmann characterized

$$\begin{array}{ccc} AbLie^{op} & \longrightarrow & Ab \\ G & \longmapsto & \pi_*^G MU \end{array}$$

III Representation stability and global spectra

<u>Recall</u>: $\widehat{Cat}^{cc}$:= categories with small colimits & cocontinuous fctrs has Lurie tensor product.

Every pointed, cocomplete category $\mathcal{C}$ is canonically left-tensored over $\mathcal{S}_*$.

$\rightsquigarrow$ $\mathcal{S}_* \times \mathcal{C} \xrightarrow[-\otimes-]{} \mathcal{C}$.

A pointed, cocomplete category $\mathcal{C}$ is <u>stable</u> if $S^1 \otimes - : \mathcal{C} \to \mathcal{C}$ is an equivalence.

Cat_{Glo}^{orb-cc} := Orb-cocomplete Glo categories and orb-cocontinuous functors.

↑ has Lurie tensor product.

A Glo-category $\mathcal{C} : \mathrm{Glo}^{op} \longrightarrow \mathrm{Cat}$ is Orb-cocomplete if

i) $\forall G \in \mathrm{Glo}$, $\mathcal{C}(G)$ is cocomplete

$\forall H \xrightarrow{\varphi} G \in \mathrm{Glo}$, $\mathcal{C}(G) \xrightarrow{\varphi^*} \mathcal{C}(H)$ is cocontinuous.

ii) $\forall$ pullbacks

$$\begin{array}{ccc} P & \xrightarrow{j} & X \\ {\scriptstyle p}\downarrow & \lrcorner & \downarrow{\scriptstyle q} \\ H & \xrightarrow{i} & G \end{array} \quad \in \mathrm{Glo}, \text{ in } \mathrm{PSh}(\mathrm{Glo}), \; q \in \mathrm{Orb}$$

$$\begin{array}{ccc} \mathcal{C}(P) & \xleftarrow{j^*} & \mathcal{C}(X) \\ {\scriptstyle p^*}\uparrow & & \uparrow{\scriptstyle q^*} \\ \mathcal{C}(H) & \xleftarrow{i^*} & \mathcal{C}(G) \end{array}$$

is horizontally left adjointable,

i.e. $\exists$ left adj'ts $i_! \dashv i^*$, $j_! \dashv j^*$ and Beck-Chevalley map

$j_! p^* \xrightarrow{\simeq} q^* i_!$ is an equivalence.

$\mathbb{S}^{orb}_*$: $G \longmapsto \mathbb{S}^G_*$ is idempotent in $\mathrm{Cat}_{\mathrm{Glo}}^{orb}$.

$\mathrm{LMod}_{\mathbb{S}^{orb}_*}(\mathrm{Cat}_{\mathrm{Glo}}^{orb}) \simeq$ pointed, Orb-cocomplete categories.

$\rightsquigarrow$ Every pointed Orb-cocomplete category is canonically left-tensored over $\mathbb{S}^{orb}_*$.

Get $- \otimes - : \mathbb{S}^G_* \times \mathcal{C}(G) \longrightarrow \mathcal{C}(G)$ $\forall$ $G \in \mathrm{Glo}$.

Def. A pointed, Orb-cocomplete Glo-category $\mathcal{C}$ is <u>representation stable</u> if for all compact Lie groups G and all finite-dimensional G-representations V,

$S^V \otimes - : \mathcal{C}(G) \longrightarrow \mathcal{C}(G)$ is an equivalence.

Proposition (Lenz-B.)

i) $Glo^{op} \ni G \xmapsto{\underline{Sp}^{gl}} Sp^{G\text{-}gl}$ G-global spectra

is the free representation-stable, Glo-cocomplete category on one generator.

ii) $Glo^{op} \ni G \xmapsto{\underline{Sp}^{G}} Sp^{G}$ genuine G spectra

is the free representation stable, Orb-cocomplete category on one generator.